

[C056/SQP182]

Advanced Higher
Mathematics
Specimen Question Paper

Time: 3 hours

NATIONAL
QUALIFICATIONS

Read carefully

1. Calculators may be used in this paper.
2. There are five Sections in this paper.

Section A assesses the compulsory units Mathematics 1 and 2

Section B assesses the optional unit Mathematics 3

Section C assesses the optional unit Statistics 1

Section D assesses the optional unit Numerical Analysis 1

Section E assesses the optional unit Mechanics 1.

Candidates must attempt **all** questions in Section A (Mathematics 1 and 2) **and one** of the following:

Section B (Mathematics 3)

Section C (Statistics 1)

Section D (Numerical Analysis 1)

Section E (Mechanics 1).

3. Full credit will be given only where the solution contains appropriate working.

Section A (Mathematics 1 and 2)

Marks

All candidates should attempt this Section.

- A1.** (a) Find partial fractions for

$$\frac{4}{x^2 - 4}. \quad 2$$

- (b) By using (a) obtain

$$\int \frac{x^2}{x^2 - 4} dx. \quad 4$$

- A2.** The performance of a prototype surface-to-air missile was measured on a horizontal test bed at the firing range and it was found that, until its fuel was exhausted, its acceleration (measured in m s^{-2}) t seconds after firing was given by

$$a = 8 + 10t - \frac{3}{4}t^2.$$

- (a) Obtain a formula for its speed, t seconds after firing. 2
(b) The missile contained enough fuel for 10 seconds. What horizontal distance would it have covered on the firing range when its fuel was exhausted? 3

- A3.** Use the substitution $x = 4 \sin t$ to evaluate the definite integral

$$\int_0^2 \frac{x+1}{\sqrt{16-x^2}} dx. \quad 5$$

- A4.** Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= -1 \cdot 1 \\ x + 3y + 2z &= 0 \cdot 9. \end{aligned} \quad 5$$

- A5.** (a) Find the derivative of y with respect to x , where y is defined as an implicit function of x by the equation

$$x^2 + xy + y^2 = 1. \quad 2$$

- (b) A curve is defined by the parametric equations

$$x = 2t + 1, \quad y = 2t(t - 1).$$

- (i) Find $\frac{dy}{dx}$ in terms of t . 2
(ii) Eliminate t to find y in terms of x . 1

A6. Let $u_1, u_2, \dots, u_n, \dots$ be an arithmetic sequence and $v_1, v_2, \dots, v_n, \dots$ be a geometric sequence. The first terms u_1 and v_1 are both equal to 45, and the third terms u_3 and v_3 are both equal to 5.

(a) Find u_{11} . 2

(b) Given that v_1, v_2, \dots is a sequence of **positive** numbers, calculate $\sum_{n=1}^{\infty} v_n$. 3

A7. Use induction to prove that

$$\sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

for all positive integers n . 5

A8. Let the function f be given by

$$f(x) = \frac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2}, \quad x \neq 2.$$

(a) The graph of $y = f(x)$ crosses the y -axis at $(0, a)$. State the value of a . 1

(b) For the graph of $f(x)$

(i) write down the equation of the vertical asymptote, 1

(ii) show algebraically that there is a non-vertical asymptote and state its equation. 3

(c) Find the coordinates and nature of the stationary point of $f(x)$. 4

(d) Show that $f(x) = 0$ has a root in the interval $-2 < x < 0$. 1

(e) Sketch the graph of $y = f(x)$. (You must include on your sketch the results obtained in the first four parts of this question.) 2

A9. Let $z = \cos \theta + i \sin \theta$.

(a) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta.$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ . 5

(b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ . 1

(c) Use your answers to (a) and (b) to express $\cos 4\theta$ in terms of $\cos \theta$ and $\sin \theta$. 1

(d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m \theta - \cos^n \theta) + p$ where k, m, n, p are integers. State the values of k, m, n, p . 4

A10. In a chemical reaction, two substances X and Y combine to form a third substance Z . Let $Q(t)$ denote the number of grams of Z formed t minutes after the reaction begins. The rate at which $Q(t)$ varies is governed by the differential equation

$$\frac{dQ}{dt} = \frac{(30 - Q)(15 - Q)}{900}.$$

- (a) Express $\frac{900}{(30 - Q)(15 - Q)}$ in partial fractions. 2
- (b) Use your answer to (a) to show that the general solution of the differential equation can be written in the form

$$A \ln \left(\frac{30 - Q}{15 - Q} \right) = t + C,$$

where A and C are constants.

State the value of A and, given that $Q(0) = 0$, find the value of C . 4

Find, correct to two decimal places,

- (i) the time taken to form 5 grams of Z , 1
- (ii) the number of grams of Z formed 45 minutes after the reaction begins. 2

[END OF SECTION A]

Candidates should now attempt ONE of the following

Section B (Mathematics 3) on Page five

Section C (Statistics 1) on Page seven

Section D (Numerical Analysis 1) on Page nine

Section E (Mechanics 1) on Page eleven.

Section B (Mathematics 3)

Marks

ONLY candidates doing the course Mathematics 1, 2 and 3 should attempt this Section.

B11. Use the Euclidean Algorithm to find integers of x, y such that

$$195x + 239y = 1. \quad 5$$

B12. The $n \times n$ matrix A satisfies the equation

$$A^2 = 5A + 3I$$

where I is the $n \times n$ identity matrix.

Show that A is invertible and express A^{-1} in the form of $pA + qI$. 2

Obtain a similar expression for A^4 . 2

B13. Use Maclaurin's theorem to write down the expansions, as far as the term in x^3 , of

(i) $\sqrt{1+x}$, where $-1 < x < 1$, and 3

(ii) $(1-x)^{-2}$, where $-1 < x < 1$. 2

B14. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

in each of the cases

(i) $f(x) = 20\cos x$ 3

(ii) $f(x) = 20\sin x$ 3

(iii) $f(x) = 20\cos x + 20\sin x$. 1

B15. (a) Show that the lines

$$L_1 : \frac{x-3}{2} = \frac{y+1}{3} = \frac{z-6}{1}$$

$$L_2 : \frac{x-3}{-1} = \frac{y-6}{2} = \frac{z-11}{2}$$

intersect, and find the point of intersection.

6

(b) Let A, B, C be the points $(2, 1, 0), (3, 3, -1), (5, 0, 2)$ respectively.

Find $\vec{AB} \times \vec{AC}$.

Hence, or otherwise, obtain the equation of the plane containing A, B and C .

5

[END OF SECTION B]

Section C (Statistics 1)

Marks

ONLY candidates doing the course Mathematics 1 and 2 and Statistics 1 should attempt this Section.

- C11.** An electronics company receives 65% of its components of a particular type from supplier A and the remainder from supplier B. Of the components supplied by A, 2% are defective while 5% of those supplied by B are defective.
- Calculate the probability that a randomly selected component, found to be defective, was supplied by A. **5**
- C12.** Records show a retailer that the number of orders for Noburn toasters during any 28-day shopping period has a Poisson distribution with mean 8.
- (a) Obtain the probability that, during a randomly selected 28-day shopping period, the number of Noburn toasters ordered exceeds 8. **2**
- (b) State the number of Noburn toasters which should be in stock at the beginning of a randomly selected 28-day shopping period in order to ensure, with at least 95% probability, that demand during the period is met from that stock. **2**
- C13.** Find the probability of obtaining at least 55 heads in 100 tosses of a normal coin. **4**
- C14.** Given a list of all secondary schools in Scotland, outline the steps involved in taking a cluster sample of Higher Mathematics pupils in 5th year. **2**
- State two reasons why it might be impractical to take a simple random sample from the target population in this case. **2**
- C15.** Of 40 fish caught in a loch, 15 were of a certain species.
- Find an approximate 95% confidence interval for the proportion of this species in the loch.
- Explain what is meant by a 95% confidence interval.
- State an assumption that you have made about the 40 fish. **6**

C16. The management team at a large hospital determined that the time taken, in minutes, to admit a patient was $N(21,4^2)$.

- (a) Calculate the probability that the mean time to admit a random sample of 10 patients is less than 17.7 minutes. **4**

A new computer system for dealing with patient records was installed at the hospital and the management team was interested in whether any reduction in the mean time to admit patients had resulted. Following the introduction of the system, a random sample of times (minutes) was as follows.

23 20 16 10 20 12 23 18 16 19

- (b) Conduct an appropriate statistical test to evaluate the evidence for a reduction in the mean waiting time. **5**

[END OF SECTION C]

Section D (Numerical Analysis 1)

Marks

ONLY candidates doing the course Mathematics 1 and 2 and Numerical Analysis 1 should attempt this Section.

D11. Obtain the Taylor polynomial of degree two for the function
 $f(x) = \ln(3x - 2), x > \frac{2}{3}$, near x . **3**

Estimate the value of $f(1.05)$ using the second degree approximation. **2**

Write down the expression for the principal truncation error in this estimate, and calculate its value. Compare the value of the principal truncation error with the actual error in the approximation. **2**

D12. The following data are available for function f .

x	1	2	3	4
$f(x)$	2.7483	2.3416	1.8409	1.2268

Use the Lagrange interpolation formula to obtain a simplified quadratic expression for $f(x)$, using the values of $f(x)$, where $x = 1, 2$ and 4 . **4**

D13. Derive the Newton forward difference formula of degree two. **5**

The following table shows values which have been obtained for a function f .

i	0	1	2	3	4
x	1.6	1.7	1.8	1.9	2.0
$f(x)$	0.826	1.203	1.609	2.042	2.503

Construct a difference table for f as far as the second differences and use the Newton forward difference formula of degree two to obtain an approximation to $f(1.63)$. **4**

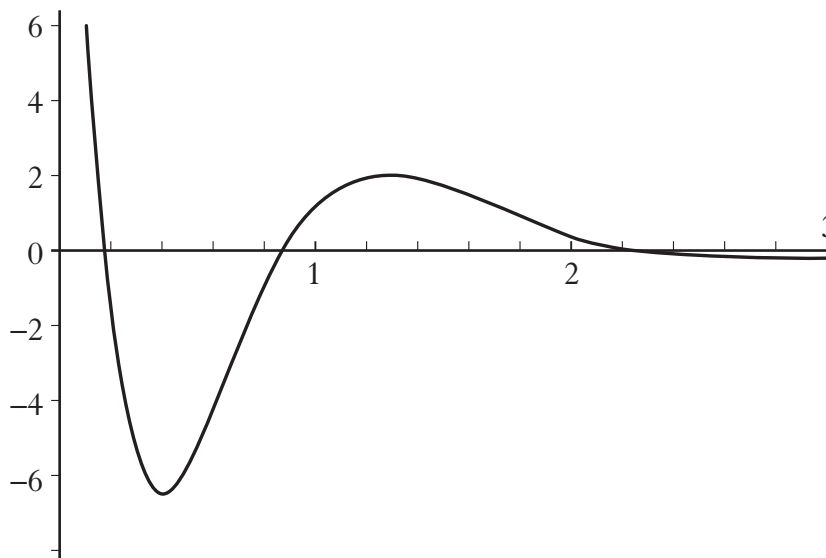
- D14.** Use Simpson's rule with two strips and the composite Simpson's rule with four strips to obtain two estimates, I_1 and I_2 respectively, for the integral

$$I = \int_1^2 x^4 e^{-2x} dx.$$

Perform the calculations using five decimal places.

4

The graph of $y = f^{(iv)}(x)$ is shown in the diagram.



The only zero of $f^{(iv)}(x)$ on the interval $[1, 2]$ occurs when $x = 1.285$ and $f^{(iv)}(1.285) = 1.903$. Use this information to obtain an estimate of the maximum truncation error in I_2 . Hence write down the value of I_2 to a suitable accuracy.

2

Establish Richardson's formula to improve the accuracy of Simpson's rule by interval halving and use it to obtain an improved estimate I_3 for I based on the values of I_1 and I_2 .

6

[END OF SECTION D]

Section E (Mechanics 1)

Marks

ONLY candidates doing the course Mathematics 1 and 2 and Mechanics 1 should attempt this Section.

Where appropriate, candidates should take the magnitude of the acceleration due to gravity as 9.8 m s^{-2} .

- E11.** A curler wishes to hit an opponent's stone. Find the minimum speed with which the curler's stone must be projected horizontally towards the opponent's stone for the curler to succeed, given that the coefficient of friction between a curling stone and the level surface of the ice is 0.04 and that the distance from the point of release of the curler's stone to the opponent's stone is 28 metres. 4
- E12.** A car travelling at 12 m s^{-1} starts to accelerate 40 metres before leaving a built-up area. The magnitude of the acceleration of the car t seconds later is given by $\frac{1}{3}(13 - 2t) \text{ m s}^{-2}$.
Find the time taken before the speed of the car first reaches 26 m s^{-1} and how far outside the built-up area the car will then be. 5
- E13.** A fishing boat sails from a harbour on a bearing of 060° at a constant speed of 26 kilometres per hour. A cargo-ship is sailing on a bearing of 030° at a constant speed of 18 kilometres per hour. The fishing boat leaves the harbour at noon, at which time the cargo ship is 10 kilometres due East of the harbour. Calculate the time at which the two vessels will be nearest to each other. 7
- E14.** A cricketer can throw a ball a **maximum** range of 60 metres to the wicket-keeper. Given that the ball is released from a point 1.5 metres above the ground and caught at the same height, calculate the speed with which it is thrown and the maximum height reached above ground level. (Air resistance may be ignored.) 6
- E15.** A packing case of mass 40 kg is to be moved through a vertical distance of 2 metres by means of a ramp of length 6 metres. The coefficient of friction between the packing case and the ramp is such that, if the packing case was placed on the ramp, it would just begin to slide down the ramp. Draw a diagram showing the forces acting on the packing case and calculate this coefficient of friction. 3
Calculate also the least force which would just move the packing case up the ramp, when the force is applied in the horizontal direction. 7

[END OF SECTION E]

[END OF QUESTION PAPER]