SCOTTISH
QUALIFICATIONS
AUTHORITY

2002 Mathematics

Advanced Higher

Finalised Marking Instructions

SECTION A (Mathematics 1 and 2)

A1.

$$\begin{array}{cccc} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 5 & 5 \end{array} \Rightarrow \begin{array}{cccc} 1 & 1 & 3 & 2 \\ & -1 & -5 & -2 \\ & -1 & -4 & -1 \end{array}$$

$$\Rightarrow \begin{array}{cccc} 1 & 1 & 3 & 2 \\ & -1 & -5 & -2 \\ & & -1 & -1 \end{array}$$

$z = 1; \quad y = -3; \quad x = 2$

Second row 1 mark
Third row 1 mark

Third row 1 mark

Values 2E1.
(available whatever method used above)

Total 5

A2.

$$i^4 + 4i^3 + 3i^2 + 4i + 2$$

$$= 1 - 4i - 3 + 4i + 2 = 0$$

Since i is a root, $-i$ must also be a root. Thus factors $(z - i)$ and $(z + i)$ give a quadratic factor $z^2 + 1$.

$$\begin{array}{r|rrrrr} & z^2 & + 4z & + 2 & & \\ \hline z^2 + 1 & z^4 & + 4z^3 & + 3z^2 & + 4z & + 2 \\ & z^4 & & + z^2 & & \\ \hline & & 4z^3 & + 2z^2 & + 4z & \\ & & 4z^3 & & + 4z & \\ \hline & & & 2z^2 & & + 2 \end{array}$$

Solving $z^2 + 4z + 2 = 0$ gives

$$z = -2 \pm \sqrt{2}.$$

1 mark for verifying and stating

1 for getting $-i$.

1 for $z^2 + 1$ is a factor.

1 for factorisation.

1 for the other two roots.

Total 5

A3.

At A, $x = -1$ so $t^2 + t - 1 = -1$ giving $t = 0$ or $t = -1$. When $t = 0$, $y = 2$. When $t = -1$, $y = 5$ so A is on the curve.

$$\frac{dx}{dt} = 2t + 1; \quad \frac{dy}{dt} = 4t - 1$$

$$\frac{dy}{dx} = \frac{4t - 1}{2t + 1}.$$

When $t = -1$, $\frac{dy}{dx} = \frac{-5}{-1} = 5$.

The equation is

$$(y - 5) = 5(x + 1)$$

$$y = 5x + 10$$

1 for solving a quadratic.

1 for the other coordinate.

1 for $\frac{dx}{dt}$ **and** $\frac{dy}{dt}$.

1 for $\frac{dy}{dx}$.

1 for the gradient is 5.

1 for an equation.

Total 6

A4.

(a) $f(x) = \sqrt{x}e^{-x} = x^{1/2}e^{-x}$

$f'(x) = \tfrac{1}{2}x^{-1/2}e^{-x} + x^{1/2}(-1)e^{-x}$

$= \dfrac{1}{2\sqrt{x}}e^{-x}(1 - 2x)$

1 method mark — 1 for first term; 1 for second term

1 for a factorised form

(b) $y = (x + 1)^2(x + 2)^{-4}$

$\log y = 2\log(x+1) - 4\log(x+2)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{x+1} - \dfrac{4}{x+2}$

$\dfrac{dy}{dx} = \left(\dfrac{2}{x+1} - \dfrac{4}{x+2}\right)y$

$a = 2;\ b = -4$

1 for taking logs and expanding

1 for differentiating

A logarithmic approach is needed.

1 for rearranging

Total 7

A5.

$\int_0^1 \ln(1 + x)\, dx$

$= \int_0^1 \ln(1 + x)\,.\,1\ dx$ — 1 for introducing the factor of 1

$= \left[x\ln(1 + x) - \int \dfrac{1}{1 + x}.x\,dx\right]_0^1$ — 1 for second term

$= \left[x\ln(1 + x) - \int\left(1 - \dfrac{1}{1 + x}\right)dx\right]_0^1$ — 2 marks for correct manipulation and integration of the second term

$= [x\ln(1 + x) - x + \ln(1 + x)]_0^1$

$= [\ln 2 - 1 + \ln 2] - [0 - 0 + 0]$ — 1 for limits

$= 2\ln 2 - 1\,[\approx 0.3863]$.

Total 5

A6.

$x + 2 = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$ — 1 for derivative

Also, $x = 2\tan\theta - 2$, so

$x^2 = 4\tan^2\theta - 8\tan\theta + 4$, giving

$x^2 + 4x + 8 = 4\tan^2\theta + 4$ — 1 for manipulation

$\int \dfrac{dx}{x^2 + 4x + 8} = \int \dfrac{2\sec^2\theta\, d\theta}{4(\tan^2\theta + 1)}$ — 1 for substitution

$= \dfrac{1}{2}\int \dfrac{\sec^2\theta\, d\theta}{\tan^2\theta + 1}$

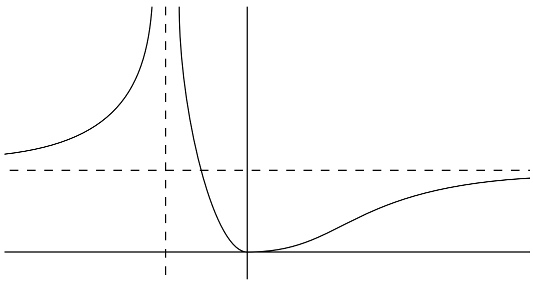
$= \dfrac{1}{2}\int d\theta$ — 1 for simplifying

$= \dfrac{1}{2}\theta + c$

$= \dfrac{1}{2}\tan^{-1}\left(\dfrac{x + 2}{2}\right) + c$ — 1 for finishing

Total 5

	Solution	Marks
A7.	When $n = 1, 4^n - 1 = 4 - 1 = 3$ so true when $n = 1$.	1 for the case $n = 1$.
	Assume $4^k - 1$ is divisible by 3.	1 for the assumption.
	Consider $4^{k+1} - 1$.	Other strategies possible.
	$4^{k+1} - 1 = 4.4^k - 1$	1 for moving to $k + 1$.
	$= (3 + 1)4^k - 1$	
	$= 3.4^k + \left(4^k - 1\right)$	1 for a correct formulation.
	Since both terms are divisble by 3 the result is true for $k + 1$.	
	Thus since true for $n = 1, 4^n - 1$ is divisible by 3 for all $n \geqslant 1$.	1 for conclusion.
		(The involvement of Σ not penalised.) Total 5

	Solution	Marks
A8.	$\dfrac{x^2}{(x+1)^2} = A + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ so	
	$x^2 = A(x+1)^2 + B(x+1) + C$	1 for valid method
	$= Ax^2 + (2A+B)x + A + B + C$	
	Hence $A = 1, B = -2$ and $C = 1$.	2E1 for the values
	(a) $y = 1 - \dfrac{2}{x+1} + \dfrac{1}{(x+1)^2}$	
	so there is a vertical asymptote $x = -1$	1 for vertical asymptote
	and a horizontal asymptote $y = 1$.	1 for horizontal asymptote
	(b) $\dfrac{dy}{dx} = \dfrac{2}{(x+1)^2} - \dfrac{2}{(x+1)^3} = 0$ at SV	1 for derivative (however obtained)
	$\Rightarrow (x+1) = 1 \Rightarrow x = 0, y = 0$	1 for solving
	$\dfrac{d^2y}{dx^2} = \dfrac{-4}{(x+1)^3} + \dfrac{6}{(x+1)^4}$	1 for justification
	$= -4 + 6$ when $x = 0$	
	Thus (0, 0) is a minimum.	1 for (0, 0) is a minimum
	(c) 	1 for asymptotes **or** 1 for each branch 1 for branches
		Total 11

A9.	(a)	
	$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-xy^2}{-x^2y} = \frac{y}{x}$$	1 mark
	$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$	1 mark
	$$\ln y = \ln x + C$$	1 mark
	$$x = 1,\ y = 2 \Rightarrow C = \ln 2$$	1 mark for evaluating C
	$$\ln y = \ln x + \ln 2$$	
	$$y = 2x$$	1 mark for formula
	(b)	
	$$\frac{dx}{dt} = -x^2(2x) = -2x^3$$	1 mark
	$$\int \frac{1}{x^3}\,dx = \int -2\,dt$$	1 mark
	$$\frac{x^{-2}}{-2} = -2t + D$$	1 mark
	$$\frac{1}{x^2} = 4t - 2D$$	
	$$t = 0,\ x = 1 \Rightarrow D = -\frac{1}{2}$$	1 mark
	$$\frac{1}{x^2} = 4t + 1$$	
	$$x = \frac{1}{\sqrt{4t + 1}}$$	1 mark Total 10

A10.	$S_n(1) = 1 + 2 + 3 + \dots + n$ $= \frac{1}{2}n(n+1)$	1 for recognising that $S_n(1)$ requires special treatment. 1 for evaluating it correctly.
	$(1-x)S_n(x) = S_n(x) - xS_n(x)$ $= 1 + 2x + 3x^2 + \dots + nx^{n-1}$ $-\left(x + 2x^2 + 3x^3 + \dots + nx^n\right)$ $= 1 + x + x^2 + \dots + x^{n-1} - nx^n$	3E1 for expanding correctly and simplifying
	$= \dfrac{1-x^n}{1-x} - nx^n.$	1 for applying the sum of a GP
	Thus $S_n(x) = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{(1-x)}$ as required.	
	$\dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \dots + \dfrac{n}{3^{n-1}} + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$ $= \left(S_n(\tfrac{1}{3}) - 1\right) + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$	1 for recognising that it relates to $S_n(\frac{1}{3})$.
	$= \dfrac{1-\frac{1}{3^n}}{(1-\frac{1}{3})^2} - \dfrac{n\frac{1}{3^n}}{1-\frac{1}{3}} - 1 + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$	1 for applying earlier result.
	$= \dfrac{9}{4}\left(1 - \dfrac{1}{3^n}\right) - \dfrac{3}{2}\cdot\dfrac{n}{3^n} - 1 + \dfrac{3}{2}\cdot\dfrac{n}{3^n}$ $= \dfrac{5}{4}\left(1 - \dfrac{1}{3^n}\right)$	
	$\lim_{n\to\infty}\left\{\dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \dots + \dfrac{n}{3^{n-1}} + \dfrac{3}{2}\cdot\dfrac{n}{3^n}\right\}$ $= \dfrac{5}{4}$	1 for obtaining the limit. Total 9

SECTION B (Mathematics 3)

B1.

(a) $\overrightarrow{AB} = 2\mathbf{i} - \mathbf{k}$; $\overrightarrow{AC} = \mathbf{i} - \mathbf{j} - 3\mathbf{k}$	1 for the two initial vectors
$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & -1 & -3 \end{vmatrix}$	1 for a cross product (Vector form acceptable.)
$= -\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$	1 for the normal vector
Equation of π_1 is of the form $-x + 5y - 2z = c$ $(1,1,0) \Rightarrow c = -1 + 5 = 4$ So an equation is $-x + 5y - 2z = 4$	1 for the equation
(b) Normals are $-\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.	1 for normals
So the angle between the planes is given by $\cos^{-1}\left(\frac{-1 + 10 - 2}{\sqrt{30}\sqrt{6}}\right)$	1 for applying the scalar product
$= \cos^{-1}\frac{7}{6\sqrt{5}}\ [\approx 58.6°]$	1 for result (must be acute) Total 7

B2.

$A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$. When $n = 1$, RHS $= \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = A$. Therefore true when $n = 1$.	1 mark for showing true when $n = 1$
Assume $A^k = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$.	1 for stating the assumption
Consider A^{k+1}. $A^{k+1} = A.A^k$	1 for considering $k + 1$
$= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$ $= \begin{pmatrix} k+2 & k+1 \\ -(k+1) & -k \end{pmatrix}$	1 for this matrix
$= \begin{pmatrix} (k+1)+1 & (k+1) \\ -(k+1) & 1-(k+1) \end{pmatrix}$	1 for obtaining final matrix
Thus if true for k then true for $k + 1$. Since true for $n = 1$, by induction, true for all $n \geqslant 1$.	1 for conclusion Total 6

	Solution	Marks
B3.	$f(x) = \ln(\cos x)$ $\quad f(0) = 0$	
	$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$ $\quad f'(0) = 0$	1 for first two derivatives
	$f''(x) = -\sec^2 x$ $\quad f''(0) = -1$	
	$f'''(x) = -2\sec^2 x \tan x$ $\quad f'''(0) = 0$	1 for third and fourth derivatives
	$f''''(x) = -4\sec^3 x \tan^2 x$ $-2\sec^4 x$ $\quad f''''(0) = -2$	1 for evaluation at 0
	$f(x) = f(0) + xf'(0) + \ldots$	1 method mark for series
	$\ln(\cos x) = 0 + 0.x - 1.\frac{x^2}{2} + 0.x - 2.\frac{x^4}{4!}$	Using series for log and cos can gain full marks.
	$= -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$	1 for an expansion Total 5
B4.	$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	1 for A
	$B = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$	1 for B
	$BA\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$	1 method for tackling a compostion
	$= \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3}x + y \\ x - \sqrt{3}y \end{pmatrix}$	
	i.e. $(x, y) \to \frac{1}{2}(\sqrt{3}x + y, x - \sqrt{3}y)$	
	so $k = \sqrt{3}$.	1 for value of k Total 4

B5.

$\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 4\cos x$	
A.E. is $m^2 + 2m + 5 = 0$	1 for auxiliary equation
$\Rightarrow m = -1 \pm 2i$	1 for roots
C. F. is $y = e^{-x}\left(A\cos 2x + B\sin 2x\right)$	1 for form of complementary function
For P.I. try $f(x) = a\cos x + b\sin x$	
$f'(x) = -a\sin x + b\cos x$	
$f''(x) = -a\cos x - b\sin x$	1 for derivatives
Thus	
$(4a + 2b)\cos x + (4b - 2a)\sin x = 4\cos x$	1 for substitution (Use of a wrong PI loses 2 of these 3 marks.)
$\Rightarrow a = 2b \Rightarrow 10b = 4$	
$\Rightarrow b = \frac{2}{5}$ and $a = \frac{4}{5}$	1 for values
$y(x) = e^{-x}\left(A\cos 2x + B\sin 2x\right) + \frac{2}{5}\left(2\cos x + \sin x\right)$	
$y(0) = 0 \Rightarrow A + \frac{4}{5} = 0 \Rightarrow A = -\frac{4}{5}$	1 for value of A
$y'(x) = e^{-x}\left(-2A\sin 2x + 2B\cos 2x\right) - e^{-x}\left(A\cos 2x + B\sin 2x\right) + \frac{2}{5}\left(\cos x - 2\sin x\right)$	1 for derivative
$y'(0) = 1 \Rightarrow 2B - A + \frac{2}{5} \Rightarrow B = -\frac{1}{10}$	1 for value of B
$y = \dfrac{e^{-x}}{10}(-8\cos 2x - \sin 2x) + \dfrac{2}{5}(2\cos x + \sin x)$	1 for final statement
	Total 10

SECTION C (Statistics 1)

C1. Quota sampling, **1**

One advantage of this method is that no sampling frame is required. **1**
One disadvantage is that it can lead to biased results. **1**

C2. P (B born same day as A and C born same day as A) = 1/7 × 1/7 = 1/49. **1**

1/365 × 1/365 = 1/133225 **1**
= 133224 to 1 (which is well away from 160 000 to 1) so disagree **1**

50000 × 1/33225 =0.375 **1**
so about once every 3 years (or 3 times in 8 years) **1**

C3. Assume that standard deviation is still 28 seconds. **1**

$$\left. \begin{array}{l} H_0 \;:\; \mu \;=\; 453 \\ H_1 \;:\; \mu \;\neq\; 453 \end{array} \right\}$$ **1**

$$z \;=\; \frac{\bar{x} \;-\; \mu}{\sigma / \sqrt{n}} \;=\; \frac{442 \;-\; 453}{28 / \sqrt{50}} \;=\; -2.78$$ **1**

$\mathrm{P}(z \;<\; -2.78) \;=\; \Phi(-2.78) \;=\; 1 \;-\; 0.9973 \;=\; 0.0027$
so that the p-value = $2 \;\times\; 0.0027 \;=\; 0.0054$ **1**

$0.0054 \;<\; 0.01$ so reject H_0 at the 1% level. **1**
i.e. there is evidence that the mean service time has changed. **1**

C4.

$$\hat{p} \;\pm\; 2.58\sqrt{\frac{\hat{p}\hat{q}}{n}}$$ **1**

$$= \;\frac{80}{250} \;\pm\; 2.58\sqrt{\frac{\frac{80}{250} \;\times\; \frac{170}{250}}{250}}$$ **1**

$$= \;0.32 \;\pm\; 0.08\;(\text{or}\;0.24 \;\rightarrow\; 0.40)$$ **1**

In the long term 99 out of 100 of intervals calculated using 99% confidence will contain the ‘true’ value of the parameter being estimated. **1**

C5. The number of shock reactions in groups of 1200 will have the binomial distribution with parameters $n = 1200$ and $p = 1/2000$. **1**

This distribution can be approximated by the Poisson distribution with parameter $1200 \times 1/2000 = 0.6$. **1**

$\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \leqslant 2)$ **1**

$= 1 - \left(e^{-0.6}.\frac{0.6^0}{0!} + e^{-0.6}.\frac{0.6^1}{1!} + e^{-0.6}.\frac{0.6^2}{2!}\right)$

$= 1 - (0.54881 + 0.32929 + 0.09879)$

$= 1 - 0.97689 = 0.023$ **1**

(arithmetical working not required as could be done on a Ti83)

C6. (a) $\mathrm{P}(X < 25)$ **1**

$= \mathrm{P}\left(Z < \frac{25 - 25.5}{0.4}\right)$

$= \mathrm{P}(Z < -1.25)$ **1**

$= 1 - \Phi(1.25) = 1 - 0.8944 = 0.1056$ **1**

(b) The distribution of the number of underweight bags is binomial **1**
with parameters $n = 40$ and $p = 0.1056$. **1**
P(No underweight bags) = ${}^{40}C_0(0.1056)^0(0.8944)^{40} = 0.0115$ **1**

(c) The mean weight of the sample of 40 bags must be less than 25 kg. **1**

$\mathrm{P}(\overline{X} < 25) = \mathrm{P}\left(Z < \frac{25 - 25.5}{0.4/\sqrt{40}}\right)$ **1**

$= \mathrm{P}(Z < -7.9)$ **1**

≈ 0 **1**

or

$T \sim N(1020, 6.4)$

and $P(T < 1000) = P(Z < -7.9)$

$\approx 0.$

SECTION D (Numerical Analysis 1)

D1.

$$L(x) = \frac{(x-1.3)(x-1.8)}{(-0.3)(-0.8)}0.758 + \frac{(x-1.0)(x-1.8)}{(0.3)(-0.5)}1.106 + \frac{(x-1.0)(x-1.3)}{(0.8)(0.5)}0.994$$

$$= (x^2 - 3.1x + 2.34)3.158 - (x^2 - 2.8x + 1.8)7.373 + (x^2 - 2.3x + 1.3)2.485$$

$$= -1.730x^2 + 5.139x - 2.651$$ **4**

D2.

$$f(x) = \sin 2x; \quad f'(x) = 2\cos 2x; \quad f''(x) = -4\sin 2x;$$

$$f'''(x) = -8\cos 2x; \quad f^{\text{iv}}(x) = 16\sin 2x$$

Taylor polynomial is

$$p\left(\frac{\pi}{4} + h\right) = \sin\frac{\pi}{2} - \frac{4h^2}{2}\sin\frac{\pi}{2} + \frac{16h^4}{24}\sin\frac{\pi}{2}$$

$$= 1 - 2h^2 + \tfrac{2}{3}h^4$$ **3**

$\sin 96° = \sin(\pi/2 + \pi/30)$ and $h = \pi/60$

Second degree approximation is

$$1 - 2\left(\frac{\pi}{60}\right)^2 = 1 - 0.0055 = 0.9945$$ **2**

Principal truncation error term is $\frac{2}{3}\left(\frac{\pi}{60}\right)^4 \approx 0.000005$

Hence second order estimate should be accurate to 4 decimal places. **2**

D3. Let quadratic through $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ be

$$y = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1).$$

Then $f_0 = A_0$; $\quad f_1 = A_0 + A_1 h$; $\quad f_2 = A_0 + 2A_1 h + 2A_2 h^2$

and so $\quad A_1 = \dfrac{f_1 - f_0}{h} = \dfrac{\Delta f_0}{h}$; $\quad A_2 = \dfrac{f_2 - 2f_1 + f_0}{2h^2} = \dfrac{\Delta^2 f_0}{2h^2}$.

Thus $y = f_0 + \dfrac{x - x_0}{h}\Delta f_0 + \dfrac{(x - x_0)(x - x_1)}{2h^2}\Delta^2 f_0$.

Setting $x = x_0 + ph$, where $0 < p < 1$, gives

$$y = f_0 + p\Delta f_0 + \tfrac{1}{2}p(p-1)\Delta^2 f_0$$ **5**

(Can also be done by an operator expansion of $(1 + \Delta)^p$.)

D4.

(a) Maximum error is 8ε. i.e. $8 \times 0.0005 = 0.004$. **1**

(b) $\Delta^2 f_1 = 0.045$ **1**

(c) Third degree polynomial would be suitable (constant differences). **1**

(d) Working from $x = 3.2, p = 0.1$

$$f(0.321) = 0.459 + 0.1(0.224) + \frac{(0.1)(-0.9)}{2}(0.051) + \frac{(0.1)(-0.9)(-1.9)}{6}(0.009)$$

$$= 0.459 + 0.022 - 0.002 + 0.000 = 0.479$$ **3**

D5. (a) Simpson's rule calculation is:

x	$f(x)$	m_1	$m_1 f_1(x)$	m_2	$m_2 f_2(x)$
1	0.4657	1	0.4657	1	0.4657
1.5	0.8320			4	3.3280
2	1.1261	4	4.5044	2	2.2522
2.5	1.0984			4	4.3936
3	0.8238	1	0.8238	1	0.8238
			5.7939		11.2633

Hence $I_2 = 5.7939/3 = 1.9313$

and $I_4 = 11.2633 \times 0.5/3 = 1.8772$

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(b) Maximum truncation error $\approx 2 \times 0.324/180 = 0.0036$ **1**

Hence suitable estimate is $I_4 = 1.88$ **1**

(c) With n strips and step size $2h$, the Taylor series for expansion of an integral approximated by Simpson's rule (with principal truncation error of $O(h^2)$) is

$$I = I_n + C(2h)^4 + D(2h)^6 + \ldots = I_n + 16Ch^4 \qquad (1)$$

With $2n$ strips and step size h, we have

$$I = I_{2n} + Ch^4 + Dh^6 + \ldots \qquad (2)$$

$16 \times (2) - (1)$ gives $15I = 16I_{2n} - I_n + O(h^6)$

i.e. $I \approx (16I_{2n} - I_n)/15 = I_{2n} + (I_{2n} - I_n)/15$ **3**

$I_3 = 1.8772 + (1.8772 - 1.9313)/15 = 1.8736$

or 1.874 to suitable accuracy **1**

SECTION E (Mechanics 1)

E1. (a) From the equation of motion for the vertical motion

$$\dot{y} = V \sin 45° - gt = \frac{1}{\sqrt{2}}V - gt. \qquad \mathbf{1}$$

The shell attains its maximum height when

$$\dot{y} = 0 \Rightarrow V = \sqrt{2}\, gt = 69.3 \text{ m s}^{-1}. \qquad \mathbf{1}$$

(b) The shell hits the ground again after 10 seconds. From the equation of motion for horizontal motion

$$x = V \cos 45° t = \frac{1}{\sqrt{2}} Vt. \qquad \mathbf{1}$$

The range is

$$R = \frac{1}{\sqrt{2}} Vt \approx 490 \text{ m}. \qquad \mathbf{1}$$

E2. (a) The position of the car is

$$x_C = \tfrac{1}{2} at^2, \qquad \mathbf{1}$$

and the position of the lorry

$$x_L = Ut + \tfrac{1}{4} at^2. \qquad \mathbf{1}$$

When the car and the lorry draw level

$$x_C = x_L \qquad \mathbf{1}$$

$$\Leftrightarrow t\left(\tfrac{1}{4}at - U\right) = 0$$

$$\Leftrightarrow t = 0 \text{ or } t = \frac{4U}{a}$$

and as $t > 0$ we take $t = \dfrac{4U}{a}$. **1**

(b) When the car draws level with the lorry it has travelled

$$x_C = \frac{1}{2} a \left(\frac{4U}{a}\right)^2 = \frac{8U^2}{a}. \qquad \mathbf{1}$$

E3. (a) Resolving perpendicular to the plane

$$N + P \sin 30° = mg \cos 30° \qquad \mathbf{1}$$

$$\Rightarrow N = \sqrt{3}\, g - \tfrac{1}{2}P$$

$$= \tfrac{1}{2}\left(2\sqrt{3}\, g - P\right). \qquad \mathbf{1}$$

The frictional force is

$$F = \mu N = \tfrac{1}{4}\left(2\sqrt{3}\, g - P\right). \qquad \mathbf{1}$$

(b) Resolving parallel to the plane and using Newton II

$$P \cos 30° = mg \sin 30° + F \qquad \mathbf{1}$$

$$\Leftrightarrow \frac{\sqrt{3}}{2}P = g + \frac{1}{4}\left(2\sqrt{3}\, g - P\right)$$

$$\Leftrightarrow \tfrac{1}{2}\left(\sqrt{3} + \tfrac{1}{2}\right)P = \left(1 + \tfrac{1}{2}\sqrt{3}\right) g \qquad \mathbf{1}$$

$$\Leftrightarrow P = \frac{2\left(2 + \sqrt{3}\right) g}{\left(2\sqrt{3} + 1\right)} \approx 16{\cdot}4 \text{ N}. \qquad \mathbf{1}$$

E4. (a) Resolving forces horizontally gives

$$T_1 \cos 30° = T_2 \cos 60° \qquad \mathbf{1}$$

$$\Rightarrow \frac{\sqrt{3}}{2} T_1 = \frac{1}{2} T_2$$

$$\Rightarrow T_2 = \sqrt{3}\, T_1 > T_1. \qquad \mathbf{1}$$

(b) Resolving forces vertically and using Newton II

$$ma = T_1 \sin 30° + T_2 \sin 60° - mg \qquad \mathbf{1}$$

$$\Rightarrow \frac{1}{2} T_1 + \frac{\sqrt{3}}{2} T_2 = m(a + g) \qquad \mathbf{1}$$

$$\frac{1}{2} \frac{1}{\sqrt{3}} T_2 + \frac{\sqrt{3}}{2} T_2 = m(a + g) \qquad \mathbf{1}$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{3}} + \sqrt{3}\right) T_2 = m(a + g)$$

$$\Rightarrow T_2 = \frac{\sqrt{3}}{2} m(a + g) \qquad \mathbf{1}$$

E5. (a) Since $\mathbf{a}_A = -\frac{2}{5}t\mathbf{i}$, $\mathbf{v}_A(t) = -\frac{1}{5}t^2\mathbf{i} + \mathbf{c}$.

Since $\mathbf{v}_A(0) = 10\mathbf{i}$, we have $\mathbf{c} = 10\mathbf{i}$ so

$$\mathbf{v}_A(t) = \left(10 - \tfrac{1}{5}t^2\right)\mathbf{i} \qquad \mathbf{1}$$

Integrating again gives

$$\mathbf{r}_A(t) = \left(10t - \tfrac{1}{15}t^3\right)\mathbf{i} + \mathbf{c}_2$$

but since $\mathbf{r}(0) = \mathbf{0}$ then $\mathbf{c}_2 = \mathbf{0}$ and

$$\mathbf{r}_A(t) = \frac{t}{15}\left(150 - t^2\right)\mathbf{i} \qquad \mathbf{1}$$

(b)(i)

$$\dot{\mathbf{r}}_B = \tfrac{1}{15}\left\{75 - 3t^2\right\}\mathbf{i} = \mathbf{0} \text{ when} \qquad \mathbf{1}$$

$$3t^2 = 75. \qquad \mathbf{1}$$

$$t = 5$$

When $t = 5$ $\quad \mathbf{r}_B = \tfrac{1}{15}\left\{45 + 375 - 125\right\}\mathbf{i} + 4\mathbf{j}$

$$= \tfrac{59}{3}\mathbf{i} + 4\mathbf{j}. \qquad \mathbf{1}$$

So the distance from the origin = $\sqrt{\left(\frac{59}{3}\right)^2 + 4^2} \approx 20.1$ m $\qquad \mathbf{1}$

(ii)

$$\mathbf{r}_A - \mathbf{r}_B = \tfrac{1}{15}t\left(150 - t^2\right)\mathbf{i} - \tfrac{1}{15}\left(45 + 75t - t^3\right)\mathbf{j} - 4\mathbf{j}$$

$$= \tfrac{1}{15}\left(75t - 45\right)\mathbf{i} - 4\mathbf{j} = \left(5t - 3\right)\mathbf{i} - 4\mathbf{j} \qquad \mathbf{1}$$

$$|\mathbf{r}_A - \mathbf{r}_B|^2 = (5t - 3)^2 + 16 \qquad \mathbf{1}$$

To find the minimum value

$$\frac{d}{dt}\left(|\mathbf{r}_A - \mathbf{r}_B|^2\right) = 2(5t - 3) \times 5 = 0 \qquad \mathbf{1}$$

so the minimum occurs when $t = \frac{3}{5}$. $\qquad \mathbf{1}$

The minimum distance is then $\sqrt{16} = 4$ m. $\qquad \mathbf{1}$

[END OF MARKING INSTRUCTIONS]